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THE RESPONSE OF THE UPPER ATMOSPHERE TO PERTURBATIONS FROM DIFFUSIVE EQUILIBRIUM

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ABSTRACT

It is generally assumed that in the atmosphere above 120 km the deviations from diffusive equilibrium are small, though a minor constituent may show some deviation. The response of such a constituent to perturbations from diffusive equilibrium is qualitatively analyzed. It is shown that the magnitude of these deviations is mainly determined by the characteristic diffusion time. For a time-dependent perturbation the ratio of the characteristic time of the perturbation to the characteristics diffusion time is an important parameter.

THE RESPONSE OF THE UPPER ATMOSPHERE TO PERTURBATIONS FROM DIFFUSIVE EQUILIBRIUM

INTRODUCTION

The atmosphere above a height of 120 km is generally considered to be close to diffusive equilibrium. Most atmospheric models are based on this assumption. Especially, the very good description of atmospheric densities given by the semi-empiric Jacchia model (Jacchia, 1971) that is based on diffusive equilibrium, shows that diffusive equilibrium of the atmospheric constituents, save possibly hydrogen and helium, is a close approximation to the true state of the atmosphere. On the other hand, some of the more detailed atmospheric structure may be related to deviations from diffusive equilibrium (Mayr and Volland, 1971).

The following processes must be considered as possible sources of deviations from diffusive equilibrium:

1. Eddy diffusion
2. Dissociation and recombination of oxygen
3. Horizontal flow having a non-zero divergence.

In this paper we shall obtain qualitative results concerning the response of a minor constituent to a perturbation from diffusive equilibrium. While for a purely qualitative analysis the nature of the source of the deviation has no particular importance, it should be remarked that for the atmosphere above

120 km both eddy diffusion and recombination of atomic oxygen may be disregarded with little loss of accuracy.

For the recombination of atomic oxygen this is immediately apparent by a consideration of the cross section of the process. Regarding eddy diffusion we have to compare the eddy diffusion coefficient with the molecular diffusion coefficient. The ratio of the eddy diffusion coefficient to the molecular diffusion coefficient is at 120 km 0.1 and at 160 km 0.01, if a height independent eddy diffusion coefficient is assumed. However, this height-independency cannot be maintained (Shimazaki, 1971), and a rather sharp decrease by one order of magnitude near a height of 105 km is probable. Taking this behaviour into account, no eddy diffusion above 120 km, or even a slightly lower altitude, need be considered.

In this paper we are only concerned with qualitative results, therefore we may use a simplified atmospheric model. Its properties are:

1. The major constituent is in diffusive equilibrium and at rest.
2. The minor constituent is in diffusive equilibrium at the upper boundary.
3. The temperature profile is given.

BASIC EQUATIONS AND DEFINITIONS

Chapman and Cowling's diffusion equation for the vertical diffusion velocity v_d of two atmospheric constituents reads in their original notation

$$v_d = v_1 - v_2 = -\frac{n^2}{n_1 n_2} D_{12} \left(\frac{\partial n_{10}}{\partial z} + \frac{n_1 n_2 (m_2 - m_1)}{n\rho} \frac{\partial \ell n p}{\partial z} - \frac{\rho_1 \rho_2 (F_1 - F_2)}{\rho p} \right) \qquad (1)$$

Here v_i are the constituent velocities, either absolute or relative to the center of mass, D_{12} the binary diffusion coefficient, n_i the number densities and n their sum, n_{10} the ratio n_1/n, m_i the molecular masses, ρ the density, p the total pressure and F_i the external forces. For the constituents of the neutral atmosphere the latter are only the gravitational attraction and therefore equal.

The first approximation of the diffusion coefficient D_{12} as given by Chapman and Cowling is

$$D_{12} = \frac{3}{8\sigma_{12}^2 (n_1 + n_2)} \left\{ \frac{kT(m_1 + m_2)}{2\pi m_1 m_2} \right\}^{1/2} \text{cm}^2 \text{ sec}^{-1} \tag{2}$$

where T is the temperature and σ_{12} the effective collision diameter. A rearrangement of (1) yields the form

$$v_i = -D_{12} \left(\frac{1}{n_1} \frac{\partial n_1}{\partial z} + \frac{1}{H_1} + \frac{1}{H_1 g} \frac{Dv_0}{Dt} - \left(\frac{1}{n_2} \frac{\partial n_2}{\partial z} + \frac{1}{H_2} + \frac{1}{H_2 g} \frac{Dv_0}{Dt} \right) \right) \tag{3}$$

where v_0 is the center of mass motion and H_i the scale heights. When the inertial forces are neglected compared to g and use is made of the assumption that the major constituent is in diffusive equilibrium, the simpler form

$$v_d = - D_{12} \left(\frac{1}{n_1} \frac{\partial n_1}{\partial z} + \frac{1}{H_1} + \frac{1}{T} \frac{\partial T}{\partial z} \right) \tag{4}$$

results.

For a constituent that is in diffusive equilibrium the expression in brackets of (4) vanishes and accordingly the diffusion velocities are zero.

The value of the bracketed expression for a constituent that is not in diffusive equilibrium will indicate the degree of its deviation from diffusive equilibrium. A dimensionless quantity d that will be a measure of the deviation will be defined by

$$d = -\frac{v_d H}{D} = H\left(\frac{1}{n}\frac{\partial n}{\partial z} + \frac{1}{H} + \frac{1}{T}\frac{\partial T}{\partial z}\right) \tag{5}$$

where the indices were dropped for simpler notation.

CONSIDERATIONS OF DIMENSIONS

Already from considerations of dimensions we may deduce possible expressions for the characteristic time scales in the atmosphere. The quantities that appear in atmospheric dynamics are for a single constituent atmosphere k, T, H, m, g, n and the collision diameter σ^2. From these quantities only few expressions having the dimension of time may be formed. If k, T or m appear in such a combination then they must appear as kT/m because in the result neither temperature nor mass may appear. But kT/m equals Hg and therefore we may limit ourselves to combinations of H, g, n and σ^2. Of these only g includes a time component, it must therefore be in the combination with an exponent of -1/2. As exponents higher than 3 or 4 for the other quantities are improbable, only a rather narrow choice of combinations that have the dimension of time is available. Two such combinations suggest themselves: $\sqrt{H/g}$ and $(\sqrt{Hg}\,\sigma^2 n)^{-1}$. The first will be the characteristic time required for a gas

to return to barometric equilibrium if a large scale deviation from equilibrium exists, the second will be the characteristic life-time of microscopic deviations that are equalized by molecular collisions.

In diffusion processes two atmpspheric constituents take part and we have to add D_{12} to the other quantities in order to obtain a time-like combination. The ratio of the molecular masses, i.e. the scale heights, may also appear as a dimensionless coefficient. D_{12} has the dimension of $cm^2\ sec^{-1}$ and is in itself a combination of H, g, n and σ^2. Again no unique determination of the characteristic time from considerations of dimensions seems possible, but H^2/D suggests itself as the simplest physically plausible form. Here the gravitational force, the temperature and the collision diameter have all been absorbed in D_{12}.

EQUALIZATION OF AN INITIAL PERTURBATION FROM DIFFUSIVE EQUILIBRIUM

In the following we shall use the subscript m for the major constituent, i for the minor constituent, o for a time independent component, b for the initial distribution and primes to indicate partial differentiation with respect to the vertical coordinate.

For the major constituent we have

$$n_m' + \frac{n_m}{H} + \frac{n_m}{T}\frac{\partial T}{\partial z} = 0 \qquad (6)$$

For the minor constituent we assume an initial deviation from diffusive equilibrium. In the absence of a continuous perturbation the minor constituent

will finally return to diffusive equilibrium and have then the density distribution n_d where

$$n_d' + \frac{n_d}{H_i} + \frac{n_d}{T}\frac{\partial T}{\partial z} = 0 \tag{7}$$

The density of the minor constituent may be written as

$$n(z,t) = n_d(z) + n_b(z)\, e^{-\alpha t} \tag{8}$$

where n_b is the part of the initial distribution that is not in diffusive equilibrium. n_b may have positive or negative sign. As we are only concerned with qualitative results we shall assume the atmosphere to be isothermal. This simplification will not affect the order of magnitude of the results, but will simplify the analytic treatment.

Perturbations that have a vertical extension of one scale height will be considered. We have therefore $z_1 - z_0 \approx H$ where z_1 is the upper boundary and z_0 the lower boundary. At the upper boundary z_1 the minor constituents will always be in diffusive equilibrium and have a density equal to the final density. In order to realize this model we define an effective scale height H_e

$$H_e = H + \Delta H = H\left(1 + \frac{\Delta H}{H}\right) = H(1+\beta) \tag{9}$$

with the properties

$$\Delta H\,(z_1) = 0$$

$$\beta(z_1) = 0 \tag{10}$$

$$\beta = \beta_b \frac{(z - z_1)}{H} e^{-\alpha t} \tag{11}$$

$$n' + \frac{n}{H_e} = 0 \tag{12}$$

This definition assures the gradual disappearance of the deviation with time.

We obtain the partial time-derivative and the diffusion velocity

$$\frac{\partial n}{\partial t} = -\alpha n_b e^{-\alpha t} \tag{13}$$

$$n v_d = -D\left(n' + \frac{n}{H}\right) = -D\left(n_b' + \frac{n_b}{H}\right) e^{-\alpha t} \tag{14}$$

Substitution in the continuity equation

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial z}(n v_d) = 0 \tag{15}$$

gives

$$-\alpha n_b = \frac{\partial}{\partial z}\left(D\left(n_b' + \frac{n_b}{H}\right)\right) \tag{16}$$

But

$$o = n' + \frac{n}{H_e} = n' + \frac{n}{H} - \frac{n\beta}{H} = n_d' + \frac{n_d}{H} + \left(n_b' + \frac{n_b}{H}\right) e^{-\alpha t} - \frac{n\beta}{H} \qquad (17)$$

therefore

$$\left(n_b' + \frac{n_b}{H}\right) e^{-\alpha t} = \frac{n\beta}{H} = \frac{n_d\beta}{H} + \frac{n_b\beta}{H} e^{-\alpha t} \qquad (18)$$

The last term of (18) may be neglected as it is the product of small quantities.

From (16)

$$-\alpha n_b = \frac{\partial}{\partial z}\left(D \frac{n_d\beta_0}{H} \frac{(z - z_1)}{H}\right) \qquad (19)$$

We differentiate each term of the product on the right hand side of (19) and substitute after the differentiation the boundary values, i.e. $z-z_1 = -H$. From (2) it is evident that

$$\frac{\partial D}{\partial z} \approx \frac{D}{H_m} \qquad (20)$$

This yields

$$-\alpha n_b = -\frac{D}{HH_m} n_d\beta_0 + \frac{D n_d\beta_0}{H^2} + \frac{D n_d\beta_0}{H^2} \qquad (21)$$

All three terms are of the order of magnitude $D\, n_d\beta_0/H^2$. The sum of the first two terms may introduce a numerical factor of the order of unity in our result. Disregarding this factor we have

$$-\alpha = \frac{D}{H^2} \frac{n_d \beta_0}{n_b} \tag{22}$$

The positive sign of α is assured by the negative sign of β_0/n_b. The magnitude of the coefficient $n_d\beta_0/n_b$ is determined from the solution of (7) and (12) where the integration constants are adjusted to yield equal densities at the upper boundary. Using this relation the coefficient becomes

$$\frac{n_d \beta_0}{n_b} = - \frac{\beta_0}{1 - \exp(-\beta_0)} \tag{23}$$

and is therefore close to unity for small β_0.

We may according to the above estimate define as the characteristic time of diffusion in the upper atmosphere $t_d = H^2/D$ sec.

We have obtained the result that after H^2/D sec the deviation from diffusive equilibrium has fallen to 1/e of its initial value.

Some confusion may arise from the proportionality of the diffusion velocities to the diffusion coefficient as expressed in the diffusion equation (1). This seems to indicate an increase of diffusion velocities with height. This is only true for the initial diffusion velocities that arise from a instantaneous perturbation. As the diffusion velocities decrease exponentially with time with an exponent $-D/H^2$ after a time span of the order of the diffusion time, the velocities at the higher altitude will be lower than at the lower altitude.

Figure 1 shows the height profile of D and t_d as derived from the Jacchia model for $T_\infty = 1000°$ and atomic oxygen and nitrogen as the constituents.

CONTINUOUS PERTURBATIONS FROM DIFFUSIVE EQUILIBRIUM

Deviations from diffusive equilibrium that arise from a continuous source will now be considered. Such sources of deviation are the horizontal divergence of the horizontal particle flux or the production and loss term of particles. First the case of time-independent perturbations will be treated. This corresponds to the diurnally averaged deviations caused by the diurnal average of the above sources. While the diurnal average of the horizontal divergence is probably small compared to its instantaneous values, this is not the case for the production of atomic oxygen which has an average value comparable to its instantaneous values as recombination above 120 km is negligible. According to Shimazaki's results the production and loss term is of the order of 10^{-6} n sec^{-1} cm^{-3} in the thermosphere above 120 km. We shall always assume that the perturbation is proportional to the number density, an assumption that is self-evident for the horizontal divergence as well as the production and loss term.

We shall assume the perturbation to be F(z) n sec^{-1} cm^{-3}, where F(z) will be only slightly height dependent.

The continuity equation for the minor constituent for the time-independent mode becomes

$$\frac{\partial}{\partial z} (v_d n)_0 = F(z)\, n_0 \tag{24}$$

or

$$v_d = -\frac{1}{n_0} \int_z^{z_1} F(z)\, n_0\, dz \approx \bar{F} H \tag{25}$$

where $\bar{F}$ is a height average of F(z) over one scale height. The deviation d is given by (5). Accordingly

$$d = -\frac{v_d H}{D} = -\frac{\bar{F} H^2}{D} \tag{26}$$

We have therefore the result that the magnitude of the deviation from diffusive equilibrium is proportional to the characteristic diffusion time and the magnitude of the perturbation.

TIME DEPENDENT PERTURBATIONS

In the case of time dependent perturbations the deviation from diffusive equilibrium will be dependent on the characteristic diffusion time and the characteristic time of the perturbation. We shall assume a perturbation of the form

$$F_A(z, t) = F(z)\, n e^{i\omega t} \tag{27}$$

The density will be given by

$$n = n_0 (1 + d e^{i\omega t}) \tag{28}$$

A similar treatment as in the time-independent case yields

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial z}(v_d n) = F(z)\, n\, e^{i\omega t} \tag{29}$$

$$i w n_0 d e^{i\omega t} - \frac{\partial}{\partial z}\left(D\left(n' + \frac{n}{H}\right)\right) = F(z)\, n\, e^{i\omega t} \tag{30}$$

$$n' + \frac{n}{H} = \left((n_0 d)' + \frac{n_0 d}{H}\right) e^{i\omega t} = n_0 d' e^{i\omega t} \tag{31}$$

because

$$n_0' = -\frac{n_0}{H} \tag{32}$$

Integration of (30) results in

$$i\omega \int_{z_0}^{z_1} dn_0\, dz - D\, n_0 d' = \bar{F}\, n_0 H \tag{33}$$

where the right hand side was linearized as d is assumed to be small.

$$d' - \frac{i\omega H}{D} d = -\frac{\bar{F} H}{D} \tag{34}$$

We define a ratio r of the characteristic times as

$$r = \frac{H^2}{D} \Big/ \frac{1}{\omega} = \frac{t_d}{t_p} = \frac{\omega H^2}{D} \tag{35}$$

and obtain

$$d' - \frac{ird}{H} = -\frac{\bar{F}t_d}{H} \tag{36}$$

If $r \ll 1$ the equation reduces to

$$d' = -\frac{\bar{F}t_d}{H} \tag{37}$$

and integration over a scale height yields

$$d = Ft_d \tag{38}$$

This expresses again that the deviation is proportional to the amplitude of the perturbation and the characteristic diffusion time.

In case the ratio r has a magnitude that approaches, or exceeds, unity, we may not neglect the second term of (36). In this case the solution is given by

$$d = \exp\left(-i\int_z^{z_1}\frac{r}{H}\,dz\right)\int_z^{z_1}\frac{F(z)t_d}{H}\exp\left(i\int_z^{z_1}\frac{r}{H}\,dz\right)dz \tag{39}$$

where we have made use of the vanishing deviation at the upper boundary. In this case, which is applicable to the height region below about 160 km, the resulting deviation depends on the ratio of the characteristic times and the characteristic diffusion time.

We may solve (39) formally by observing that both r and t_d decrease exponentially with altitude. The result will be a deviation that has an altitude

dependent phase factor and an amplitude that is proportional to the perturbation amplitude F(z) and the diffusion time H^2/D.

$$d \approx \bar{F} t_d \, e^{if(r)} = \bar{F} t_d \, e^{iwf_1(z)} \tag{40}$$

RESULTS

1. The upper atmosphere responds to perturbations from diffusive equilibrium with a character time H^2/D.

2. Perturbations that extends over a height range of the order of a scale height are damped with a factor $\exp(-D/H^2 \cdot t)$.

3. Deviations from diffusive equilibrium that arise due to continuously acting sources are proportional to the amplitude of the perturbation and the characteristic diffusion time.

4. In the height region where the characteristic time of the perturbation is of the order of the characteristic diffusion time, or of smaller order, a height dependent phase shift between the deviation and the source is introduced.

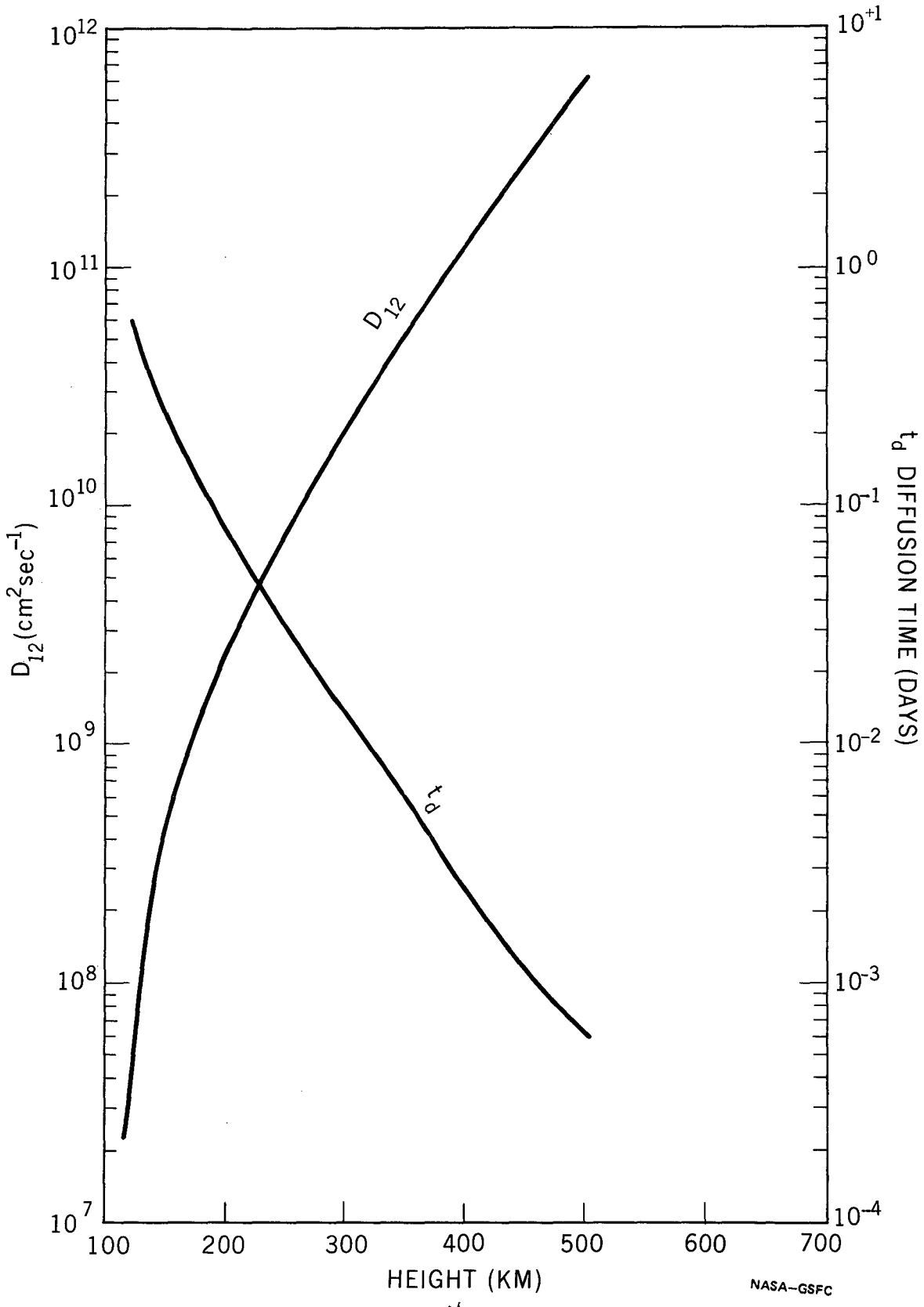
REFERENCES

Chapman, S. and Cowling, T. G., The mathematical theory of non-uniform gases, Cambridge University Press, 1953.

Jacchia, L., Smithsonian Ast. Obser. Special Report 332, 1971.

Mayr, H. G., and Volland, H., J. Geophys. Res. 77, 2359, 1972.

Shimazaki, T., J. Atmos. Terr. Phys. 33, 1382, 1971.


10^{12}
10^{11}
10^{10}
10^{9}
10^{8}
10^{7}
$D_{12}(cm^2sec^{-1})$
10^{+1}
10^{0}
10^{-1}
10^{-2}
10^{-3}
10^{-4}
t_d DIFFUSION TIME (DAYS)
D_{12}
t_d
100
200
300
400
500
600
700
HEIGHT (KM)
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